

DUALITY AND TOPOLOGICAL QUANTUM FIELD THEORY*

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ABSTRACT

We present a summary of the applications of duality to Donaldson-Witten theory and its generalizations. Special emphasis is made on the computation of Donaldson invariants in terms of Seiberg-Witten invariants using recent results in $N = 2$ supersymmetric gauge theory. A brief account on the invariants obtained in the theory of non-abelian monopoles is also presented.

Topological quantum field theories^{35,11,13} (TQFTs) have provided very powerful tools to address problems related to the topology of low-dimensional manifolds. They involve field theory representations of some of the most recent topological invariants studied on these manifolds. In three dimensions Chern-Simons gauge theory³⁷ has led to a description of polynomial invariants for knots and links which generalize the Jones polynomial¹⁷. In four dimensions, the TQFT known as Donaldson-Witten theory³⁵ has given a new way to look at Donaldson invariants¹⁴. The connection between these theories and these two types of invariants was achieved studying the corresponding quantum field theories in their non-perturbative and perturbative regimes, respectively. It was clear from the first works in the field that an analysis of these theories in other regimes would certainly provide new points of view on these invariants.

Chern-Simons gauge theory was studied from a perturbative point of view^{15,8} soon after its formulation by Witten in 1988. The perturbative series provided an infinite sequence of numerical knot invariants which were later identified with Vassiliev invariants^{10,19,9}. Vassiliev invariants were formulated from a mathematical point of view³⁴ basically at the same time. It took, however, some time until a relation between them was pointed out. Nevertheless, Chern-Simons gauge theory has provided an integral representation of at least a subset of Vassiliev invariants². Furthermore, the richness of quantum field theory is being exploited to extend these invariants for the case of links³ and most likely, through the analysis of the theory in different gauges, it will provide alternative representations.

In four dimensions, the analysis of Donaldson-Witten theory in a regime different

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than the perturbative one seemed a rather unsurmountable problem before 1994. However, in that year Seiberg and Witten unraveled the strong-coupling behavior of $N = 2$ supersymmetric gauge theory using arguments based on duality³². Their results could certainly be applied to Donaldson-Witten theory since this theory can be regarded as a twisted form of $N = 2$ supersymmetric gauge theory. Indeed this was done by Witten³⁹ in the fall of 1994 obtaining an unexpected new representation for Donaldson invariants. In this representation Donaldson invariants are expressed in terms of a new type of invariants now known as Seiberg-Witten invariants. These invariants turn out to be simpler than Donaldson invariants and, presumably, more powerful. They have certainly opened a new point of view on the topology of four-manifolds.

In this paper we will concentrate on Donaldson-Witten theory. The aim is to account for the progress made after the appearance of Seiberg-Witten solution for $N = 2$ supersymmetric gauge theories. Before entering into details let us briefly describe how ideas based on duality, intrinsic in this solution, can be used in the contest of Donaldson-Witten theory.

Donaldson-Witten theory can be formulated in terms of a twisted version of $N = 2$ supersymmetric gauge theory or, from a more geometrical point of view, in terms of a representative of the Thom class of a vector bundle associated to certain moduli problem in the framework of the Mathai-Quillen formalism⁷. Twisted $N = 2$ supersymmetric theories, in general, are associated to certain moduli problems which, properly treated in the contest of the Mathai-Quillen formalism, lead to representatives of the Thom class which are the exponential of the twisted action. Both pictures of Donaldson-Witten theory have been known for some time. One important property of the resulting TQFT is that the vevs. of its observables are independent of the coupling constant. This means that these vevs. could be computed in either the strong or the weak coupling limit. The weak coupling limit analysis showed the relation of the observables of the theory to Donaldson invariants. However, in such analysis no new progress was made from the quantum field theory representation in what respect to the calculation of these invariants. The difficult problems that one has to face are similar to the ones in ordinary Donaldson theory.

In 1994 Seiberg and Witten worked out the strong coupling limit behavior of $N = 2$ supersymmetric Yang-Mills theory³². One would expect that the twisted version of the corresponding effective theory would be related to Donaldson-Witten theory. Furthermore, since in the TQFT observables are independent of the coupling constant, the weak coupling limit of the effective theory should be exact, *i.e.*, it would lead to Donaldson invariants. This is in fact what turns out to be the case. The twisted effective theory could be regarded as a TQFT *dual* to the original one. In addition, one could ask for the *dual* moduli problem associated to this dual TQFT. It turns out that this moduli space is an abelian version of the moduli space of instantons modified by the presence of chiral spinors. This space is known as the moduli space of

abelian monopoles³⁹. Being related to an abelian gauge theory this space is simpler to analyze than the moduli space of instantons. Furthermore, for a large set of four-manifolds (of simple type) only particular classes of abelian gauge fields (basic classes) contribute. For these classes the moduli space of abelian monopoles reduces to a finite set of points.

Generalizations of Donaldson-Witten theory introducing matter fields have been studied for some time^{18,1,5}. Recently, the resulting theories have been understood in the Mathai-Quillen formalism^{21,22} and their non-perturbative analysis has been carried out, at least for the gauge group $SU(2)$ and one multiplet of matter in the fundamental representation. From a perturbative point of view the associated moduli space corresponds to non-abelian monopoles. The set of observables of this theory has a structure which is similar to the one in standard Donaldson-Witten theory and, although they are based in a different moduli space, it turns out that they can also be expressed in terms of Seiberg-Witten invariants. There seems to exist a set of types of topological invariants which can be expressed²³ in terms of these invariants.

After a brief introduction to TQFT in the first section we will describe in the second section how Donaldson invariants are computed in Donaldson-Witten theory using recent results in $N = 2$ supersymmetric gauge theory. In section three we present a brief description of the theory of non-abelian monopoles as well as some concluding remarks.

1. Topological Quantum Field Theory

To be specific, we will restrict ourselves to a particular set of TQFTs, namely those with a gauge symmetry and a BRST operator Q . These theories are then of cohomological type and are usually constructed starting from a supersymmetric gauge theory. Both Donaldson-Witten theory and the Seiberg-Witten theory of abelian monopoles are theories of this type, as well as the non-abelian monopole theory. The two key points in these kinds of theories are the mathematical interpretation associated to them and the relation to the underlying physical model. We will begin with a rather general exposition of the first aspect, and then we will discuss the twisting procedure which makes the connection with $N = 2$ supersymmetric gauge theories. More details can be found in references^{11,13,27}.

Topological quantum gauge theories of the cohomological type are characterized by the following data. First of all we have a configuration space \mathcal{X} , whose elements are fields ϕ_i defined on some Riemannian manifold M . Among these fields there is the gauge connection associated to a principal G -bundle P over M , A . The group of gauge transformations \mathcal{G} acts on \mathcal{X} according to the different gauge-structure of the fields. For instance, on the gauge connections the gauge transformations are the standard ones, and if there are matter fields in \mathcal{X} , \mathcal{G} acts on them according to the representation of the gauge group we choose. In this case, there is also a representation

bundle E on M associated to P . Notice that in general the configuration space \mathcal{X} is infinite-dimensional.

The next ingredient in the construction is an (infinite-dimensional) vector space \mathcal{F} also with an action of the group of gauge transformations. This vector space corresponds, as we will see in a moment, to the quantum numbers of the field equations. One can then consider the configuration space \mathcal{X} as a principal \mathcal{G} -bundle and \mathcal{F} as a representation space and construct the associated vector bundle,

$$\mathcal{E} = \mathcal{X} \times_{\mathcal{G}} \mathcal{F}, \quad (1)$$

which will be the fundamental geometrical object in this class of theories. The last ingredient is a \mathcal{G} -equivariant map,

$$s : \mathcal{X} \rightarrow \mathcal{F}, \quad s(g \cdot \phi_i) = g \cdot s(\phi_i), \quad (2)$$

which descends to a section $\hat{s} : \mathcal{X}/\mathcal{G} \rightarrow \mathcal{E}$. The zero locus of the section \hat{s} is the *moduli space* \mathcal{M} associated to the problem, and its elements are precisely the fields that verify the equation $s(\phi_i) = 0$, modulo gauge transformations. This is already a familiar object in field theory. Instanton equations, the Bogomolny monopole equations or the Dirac equation can be all reinterpreted in this framework. Although we started from infinite dimensional spaces, the moduli space \mathcal{M} will be finite-dimensional provided that the map s verifies some conditions (ellipticity), and in this case the dimension of the moduli space can be generically obtained from an index theorem.

The topological charge can be understood in this context as follows: as both \mathcal{X} , \mathcal{F} have a \mathcal{G} action, there is a correspondence between elements in the Lie algebra of \mathcal{G} , $\text{Lie}(\mathcal{G})$, and vector fields on \mathcal{X} , \mathcal{F} , given by a map C . This map, geometrically, gives the fundamental vector fields on the principal \mathcal{G} - bundle \mathcal{X} , and physically corresponds to the gauge slices in the space of fields. One can then construct the operator,

$$Q = d - \phi^a \iota(C(T_a)), \quad (3)$$

where d is the de Rham operator, T_a is a basis of the Lie algebra of G , and ϕ^a are the components of an arbitrary field $\phi \in \text{Lie}(\mathcal{G})$ w.r.t. the basis T_a (not to be confused with the general fields introduced before). ι refers as usual to the inner product with a vector field. This operator gives precisely the \mathcal{G} -equivariant cohomology of the configuration space \mathcal{X} , and the cohomology classes thus obtained are in one-to-one correspondence with the cohomology of \mathcal{X}/\mathcal{G} . These cohomology classes are called the *observables* of the TQFT.

What can one compute in this context? The first quantity of topological interest arises when the dimension of the moduli space is zero. In the finite-dimensional case this happens (for generic, transversal sections) when the rank of \mathcal{E} equals the dimension of the base space \mathcal{X}/\mathcal{G} . In this case, one can compute the Euler characteristic of the vector bundle \mathcal{E} by integrating its Euler class $\chi(\mathcal{E})$ on the base manifold. The

Euler class can be obtained as the pullback of the Thom class of \mathcal{E} , $\Phi(\mathcal{E})$ using the section \hat{s} . One then has for the Euler characteristic the expression,

$$Z = \int_{\mathcal{X}/\mathcal{G}} [d\phi_i] \hat{s}^*(\Phi(\mathcal{E})), \quad (4)$$

where the fields in the measure are defined modulo gauge transformations. If the dimension of the moduli space is non-zero, there are other interesting quantities, the intersection numbers in the moduli space \mathcal{M} . Consider then a basis for the cohomology ring of \mathcal{X}/\mathcal{G} , given by the observables of the theory \mathcal{O}_a . If we group all the observables in a generating function for the intersection numbers, we get:

$$\langle \exp(\sum_a \alpha_a \mathcal{O}_a) \rangle = \int_{\mathcal{M}} \exp(\sum_a \alpha_a i^* \mathcal{O}_a), \quad (5)$$

where $i : \mathcal{M} \hookrightarrow \mathcal{X}/\mathcal{G}$ is the inclusion of the moduli space into the configuration space modulo gauge transformations. But it is a standard result in algebraic topology that $\chi(\mathcal{E})$ is the Poincaré dual of the zero locus $\mathcal{M} = \hat{s}^{-1}(0)$. One then obtains

$$\langle \exp(\sum_a \alpha_a \mathcal{O}_a) \rangle = \int_{\mathcal{X}/\mathcal{G}} [d\phi_i] \chi(\mathcal{E}) \wedge \exp(\sum_a \alpha_a \mathcal{O}_a). \quad (6)$$

From this expression one sees that, in the expansion of the exponential, there will be contributions only for those products of observables whose total degree (as differential forms) equals the dimension of \mathcal{M} . This is a selection rule for the correlation functions of the observables, and has a Field Theory description: the degree of the forms is the quantum number associated to an $U(1)_{\mathcal{R}}$ symmetry, and the selection rule is the ‘t Hooft condition.

To make the connection with Field Theory, we still need more ingredients. Although we won’t give the details here, the crucial step involves the introduction of a particular representative for the Thom class due to Mathai and Quillen²⁸. The Mathai-Quillen formalism has been crucial to elucidate the geometric structure of Cohomological Gauge Theories, and its importance in this context was first realized by Atiyah and Jeffrey in reference⁷ and further clarified in reference¹³. Let us see how to obtain a field theory description of the topological invariants introduced above. Denote by ψ_i a set of fields associated to the fibre \mathcal{F} and to the Lie algebra of the group of gauge transformations $\text{Lie}(\mathcal{G})$. We also introduce Grassmannian fields associated to the integration of differential forms on the spaces involved. We will denote them by $\hat{\phi}_i, \hat{\psi}_i$. Physically they correspond to the superpartners of the fields ϕ_i, ψ_i . The result of ^{7,13} can be summarized in the following equation:

$$\pi^*(\chi(\mathcal{E}))\Phi(\mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}) = \int [d\psi]_i [d\hat{\psi}]_i \exp(-S_{\text{top}}[\phi_i, \psi_i, \hat{\phi}_i, \hat{\psi}_i]). \quad (7)$$

In this equation, π is the projection map of the principal \mathcal{G} - bundle \mathcal{X} , and $\Phi(\mathcal{X} \rightarrow \mathcal{X}/\mathcal{G})$ is a projection form that allows one to lift the integration on \mathcal{X}/\mathcal{G} to the

full principal bundle \mathcal{X} . $S_{\text{top}}[\phi_i, \psi_i, \hat{\phi}_i, \hat{\psi}_i]$ is the action of a certain field theory, the TQFT associated to the moduli problem. Notice that the term in the l.h.s. is a differential form on the bundle \mathcal{X} , and depends on the fields ϕ_i and its superpartners $\hat{\phi}_i$. These superpartner fields are to be understood as a basis of differential forms for the “coordinate” fields ϕ_i . Taking this into account, we can write the Euler characteristic and the intersection numbers as,

$$\begin{aligned} Z &= \int_{\mathcal{X}} [d\phi_i][d\hat{\phi}_i][d\psi_i][d\hat{\psi}_i] \exp(-S_{\text{top}}[\phi_i, \psi_i, \hat{\phi}_i, \hat{\psi}_i]), \\ \langle \exp(\sum_a \alpha_a \mathcal{O}_a) \rangle &= \int_{\mathcal{X}} [d\phi_i][d\hat{\phi}_i][d\psi_i][d\hat{\psi}_i] \exp(-S_{\text{top}}[\phi_i, \psi_i, \hat{\phi}_i, \hat{\psi}_i] + \sum_a \alpha_a \mathcal{O}_a). \end{aligned} \quad (8)$$

In this way, we already have the fundamental bridge between the moduli problem and the field theory framework: associated to a moduli problem there is a certain TQFT with action S_{top} . The Euler characteristic of the bundle involved in the moduli problem is the partition function of the TQFT, and the intersection numbers on the moduli space are the correlation functions of the observables \mathcal{O}_a . There are two important properties about the topological action appearing here, and both can be derived in the framework of the Mathai-Quillen formalism. The first one is that the bosonic part of S_{top} has the form,

$$S_{\text{top}} = ||s(\phi_i)||^2 + \cdots, \quad (9)$$

where the norm is taken w.r.t. an appropriate \mathcal{G} -invariant metric on the vector space \mathcal{F} . The second property has been crucial to understand the properties of TQFTs. The Q operator can be extended in a natural way to act on all the fields in the theory. On the configuration space \mathcal{X} it is defined as in (3). The fields $\psi_i, \hat{\psi}_i$ are fields in the vector space \mathcal{F} or in the Lie algebra $\text{Lie}(\mathcal{G})$. In the first case, Q acts as in (3) as well, and the action on the fields in the Lie algebra is just the coadjoint action. With the Q operator extended in this way, one has a BRST operator for the TQFT defined by S_{top} . As \mathcal{O}_a and S_{top} are representatives of (equivariant) cohomology classes, one has:

$$[Q, \mathcal{O}_a] = [Q, S_{\text{top}}] = 0. \quad (10)$$

Furthermore, in almost all the TQFTs of the cohomological type, one also has

$$S_{\text{top}} = \{Q, V\}, \quad (11)$$

where V is a certain functional of the fields usually called gauge fermion. The fact that S_{top} is Q -exact does not mean that S_{top} is trivial. Recall that, because of (7), the topological action is essentially a characteristic class, and these can be locally trivialized in terms of secondary characteristic classes. This also allows one to prove that characteristic classes are truly topological, and they don't depend on the metric or connection that one chooses to obtain a representative. Namely, using this local

exactness one can easily prove that the variation of a characteristic classes w.r.t. a change in the connection is just an exact form. In the same way, if one considers the change of the topological action w.r.t. to a change in the metric of the underlying manifold M , $g_{\mu\nu}$ (which generically will explicitly appear in S_{top}), one obtains from (11)

$$T_{\mu\nu} = \frac{\delta S_{\text{top}}}{\delta g^{\mu\nu}} = \{Q, \frac{\delta V}{\delta g^{\mu\nu}}\}. \quad (12)$$

The energy-momentum tensor is then a BRST commutator.

After this review of the mathematical framework of topological quantum gauge theories of the cohomological type, we will briefly address the most useful procedure to obtain them from $N = 2$ supersymmetric gauge theories: the twist. In this procedure the starting point is an $N = 2$ supersymmetric quantum field theory. The basic ingredient consists of extracting a scalar symmetry out of the $N = 2$ supersymmetry. Let us consider the case of $d = 4$. In \mathbf{R}^4 the global symmetry group when $N = 2$ supersymmetry is present is $H = SU(2)_L \otimes SU(2)_R \otimes SU(2)_I \otimes U(1)_{\mathcal{R}}$ where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group and $SU(2)_I \otimes U(1)_{\mathcal{R}}$ is the internal symmetry group. The supercharges Q_{α}^i and $\overline{Q}_{i\dot{\alpha}}$ which generate $N = 2$ supersymmetry have the following transformations under H :

$$Q_{\alpha}^i \left(\frac{1}{2}, 0, \frac{1}{2}\right)^1, \quad \overline{Q}_{i\dot{\alpha}} \left(0, \frac{1}{2}, \frac{1}{2}\right)^{-1}, \quad (13)$$

where the superindex denotes the $U(1)_{\mathcal{R}}$ charge and the numbers within parentheses the representations under each of the factors in $SU(2)_L \otimes SU(2)_R \otimes SU(2)_I$.

The twist consists of considering as the rotation group the group $\mathcal{K}' = SU(2)'_L \otimes SU(2)_R$ where $SU(2)'_L$ is the diagonal subgroup of $SU(2)_L \otimes SU(2)_I$. This implies that the isospin index i becomes a spinorial index α : $Q_{\alpha}^i \rightarrow Q_{\alpha}^{\beta}$ and $\overline{Q}_{i\dot{\beta}} \rightarrow G_{\alpha\dot{\beta}}$. Precisely the trace of Q_{α}^{β} is chosen as the generator of the scalar symmetry: $Q = Q_{\alpha}^{\alpha}$. Under the new global group $H' = \mathcal{K}' \otimes U(1)_{\mathcal{R}}$, the symmetry generators transform as:

$$G_{\alpha\dot{\beta}} \left(\frac{1}{2}, \frac{1}{2}\right)^{-1}, \quad Q_{(\alpha\beta)} (1, 0)^1, \quad Q (0, 0)^1. \quad (14)$$

Once the scalar symmetry is found we must study if, as stated in (12), the energy-momentum tensor is exact, *i.e.*, if it can be written as the transformation of some quantity under Q . The $N = 2$ supersymmetry algebra gives a necessary condition for this to hold. Notice that, after the twisting, such an algebra becomes:

$$\{Q_{\alpha}^i, \overline{Q}_{j\dot{\beta}}\} = \delta_j^i P_{\alpha\dot{\beta}} \longrightarrow \{Q, G_{\alpha\dot{\beta}}\} = P_{\alpha\dot{\beta}}, \quad (15)$$

where $P_{\alpha\dot{\beta}}$ is the momentum operator of the theory. Certainly (15) is only a necessary condition for the theory being topological. However, up to date, for all the $N = 2$

supersymmetric models whose twisting has been studied, the relation on the right hand side of (15) has become valid for the whole energy-momentum tensor.

2. Donaldson-Witten Theory

Donaldson-Witten theory was historically the first TQFT, and was constructed by Witten in reference³⁵. The geometrical framework for this model is the following. We consider a Riemannian four-manifold X together with a principal G -bundle P . The configuration space is just the space of G -connections on this bundle, \mathcal{A} . The vector space \mathcal{F} is the space of self-dual forms with values in the adjoint bundle \mathfrak{g}_P , $\Omega^{2,+}(\mathfrak{g}_P)$. The map s is given by

$$s(A) = F_A^+, \quad (16)$$

where F_A^+ is the self-dual part of the curvature of the connection A . The moduli space associated to s is then the moduli space of anti-self-dual (ASD) connections on X , *i.e.*, the moduli space of ASD G -instantons. This moduli space was considered by Atiyah, Hitchin and Singer in reference⁶ and is the building block of Donaldson theory¹⁴. Indeed, the intersection numbers defined in (5) are in this case the Donaldson invariants. We will be mainly concerned with the simplest case $G = SU(2)$. The moduli space of ASD connections has then the dimension

$$\dim \mathcal{M}_{\text{ASD}} = 8k - \frac{3}{2}(\chi + \sigma), \quad (17)$$

where k is the instanton number, and χ and σ are respectively the Euler characteristic and the signature of the four-manifold X . Although we haven't discussed the subtleties concerning the structure of moduli spaces, there is a crucial point which will be important in what follows. In order to avoid singularities in the moduli space, one has to be careful with abelian instantons that can be a solution to $F_A^+ = 0$ by embedding $U(1)$ in $SU(2)$. These abelian instantons are called the *reducible solutions* to the ASD equations. One can prove that, if the manifold X is such that $b_2^+ > 1$, there are no reducible solutions. We will assume that this is the case in what follows.

The cohomology associated to the operator Q is in this case,

$$\delta A_\mu = \psi_\mu, \quad \delta \psi = d_A \phi, \quad \delta \phi = 0, \quad (18)$$

where the Grassmannian field ψ is understood as a basis of differential forms for the configuration space \mathcal{A} , and ϕ is a the generator of the \mathcal{G} -equivariant cohomology appearing in (3). The other fields in the theory include a Grassmannian self-dual 2-form $\chi_{\mu\nu}$, with values in \mathfrak{g}_P , which is associated to the fibre \mathcal{F} . One also has Lie algebra valued fields $\lambda, \eta \in \Omega^0(X, \mathfrak{g}_P)$. To construct the cohomology ring of this model one uses the *descent procedure* introduced in³⁵. The starting point is the operator $\mathcal{O}^{(0)} = \mathcal{O}$ given by

$$\mathcal{O} = \frac{1}{8\pi^2} \text{Tr}(\phi^2), \quad (19)$$

which is a differential form of degree four on the moduli space. One recursively finds k -forms on X , $\mathcal{O}^{(k)}$, which are $4-k$ forms on the moduli space, such that the following descent equations are verified:

$$d\mathcal{O}^{(k)} = \{Q, \mathcal{O}^{(k+1)}\}. \quad (20)$$

One then finds,

$$\mathcal{O}^{(1)} = \frac{1}{4\pi^2} \text{Tr}(\phi\psi), \quad \mathcal{O}^{(2)} = \frac{1}{4\pi^2} \text{Tr}(\phi F + \frac{1}{2}\psi \wedge \psi), \quad \mathcal{O}^{(3)} = \frac{1}{4\pi^2} \text{Tr}(\psi \wedge F). \quad (21)$$

From the descent equations (20) follows that if Σ is a k -dimensional homology cycle, then,

$$I(\Sigma) = \int_{\Sigma} \mathcal{O}^{(k)}, \quad (22)$$

is in the cohomology of Q . For simply connected four-manifolds (the main focus of Donaldson theory), k -dimensional homology cycles only exist for $k = 0, 2, 4$, and in the last case the observable $I(X)$ is just the instanton number. The generators of the cohomology ring of \mathcal{A}/\mathcal{G} are then the observable \mathcal{O} and the family of observables $I(\Sigma_a)$, where the Σ_a denote a basis of the two-dimensional homology of X , and therefore $a = 1, \dots, \dim H_2(X, \mathbf{Z})$. The generating function corresponding to (5) will be in this case

$$\langle \exp(\sum_a \alpha_a I(\Sigma_a) + \mu \mathcal{O}) \rangle, \quad (23)$$

and a sum over the instanton number is understood (as usual in Field Theory, we sum in the correlation functions over all the possible topological sectors). Notice that, as the differential forms on the moduli space appearing in the generating function are of even degree, (17) must be even to have non-vanishing correlation functions.

The TQFT associated to this moduli problem can be constructed with the Mathai-Quillen formalism or by a twisting of $N = 2$ supersymmetric Yang-Mills theory. This is the point of view which will be useful to make contact with the Seiberg-Witten solution and then with the abelian monopole equations. The twisting procedure is standard, and one obtains (on-shell) the same fields we introduced before in the framework of the Mathai-Quillen formalism, as well as the same operator Q (the topological charge). The field content of $N = 2$ supersymmetric Yang-Mills theory is given by an $N = 2$ vector multiplet with the following components: a one-form connection A_μ , two Majorana spinors $\lambda_{i\alpha}$, $i = 1, 2$, a complex scalar B and an auxiliary field D_{ij} (symmetric in i and j). The scalar field has an \mathcal{R} -charge -2 , while the gluinos have \mathcal{R} -charge -1 . After the twist, the fields become precisely the ones in the Mathai-Quillen formulation. Notice that, as the twist changes the spin of the fields, all the fields in the twisted model are differential forms, hence they are well defined on any curved four-manifold X (in contrast, the untwisted $N = 2$ Yang-Mills theory has spinor fields, and then it is not well-defined on manifolds which are not

Spin). The twisted action coincides with the one obtained in the framework of the Mathai-Quillen formalism when one constructs a representative for the Thom form. This equivalence between the two formulations was first discovered in reference⁷.

As we saw in the preceding section, Donaldson invariants are described by correlation functions of the twisted $N = 2$ supersymmetric Yang-Mills theory, and because of (12) they are independent of the metric g we choose on the four-manifold X . One can think then in evaluating the path-integral in the two limits $g \rightarrow 0$ (short distances, UV regime) and $g \rightarrow \infty$ (long distances, IR regime). As $N = 2$ supersymmetric Yang-Mills theory is asymptotically free, it is weakly coupled in the UV. Indeed, the semiclassical approximation is exact because of the metric independence. In this approximation the path integral is localized on the moduli space of ASD instantons, $s(A) = 0$, and we recover the usual description of Donaldson invariants in terms of intersection numbers in the moduli space. On the other hand, if one goes to the IR limit it is necessary to deal with a strongly-coupled gauge theory. Because of this, little progress has been made in understanding Donaldson theory from the physical point of view until Seiberg and Witten found the exact low-energy description of $N = 2$ supersymmetric Yang-Mills theory for the gauge group $SU(2)$ ³². We will now summarize the main aspects of this solution, stressing the features that are relevant to Donaldson theory. For a review of the Seiberg-Witten solution, one can see reference⁴.

1) Classically, the gauge symmetry is broken by the VEV of the gauge-invariant field $u = \text{Tr}(B^2)$, and is restored at $u = 0$, where the W^\pm bosons are massless. The classical moduli space of vacua is parametrized by the complex variable u and is singular at the origin.

2) Quantum-mechanically, the $SU(2)$ symmetry is never restored and the theory stays in the Coulomb phase throughout the complex u -plane. The low-energy description is then given by an $N = 2$ abelian vector multiplet \mathcal{A} (whose scalar component will be denoted by a) and an $N = 2$ prepotential $\mathcal{F}(\mathcal{A})$.

3) There is a discrete anomaly-free symmetry $\mathbf{Z}_8 \subset U(1)_{\mathcal{R}}$ which acts as a \mathbf{Z}_2 symmetry on the u -plane (as u has \mathcal{R} -charge 4). The quantum moduli space of vacua has two singularities at $\pm\Lambda^2$ (where Λ is the dynamically generated scale of the theory), related by the \mathbf{Z}_2 symmetry. These singularities are associated to extra degrees of freedom becoming massless. At $u = \Lambda^2$ there is a massless monopole with quantum numbers $(n_m, n_e) = (1, 0)$, and at $u = -\Lambda^2$ there is a massless dyon with numbers $(1, -1)$.

4) The low-energy effective theory has an $SL(2, \mathbf{Z})$ duality which relates different descriptions. Massless states at the singularities are described by $N = 2$ matter hypermultiplets, and in each case one has to use the appropriate description of the abelian field using $SL(2, \mathbf{Z})$ transformations. For the monopole singularity one takes the magnetic, S -dual photon \mathcal{A}_D , while for the dyon one takes the dyonic photon $\mathcal{A}_D - \mathcal{A}$. The scalar component of \mathcal{A}_D , a_D , gives the monopole mass through the BPS

equation $M = \sqrt{2}|a_D|$. Therefore $a_D(\Lambda^2) = 0$. In the same way, $(a_D - a)(-\Lambda^2) = 0$

Now that we have a precise description of the IR physics of $N = 2$ supersymmetric Yang-Mills theory, we can try to compute the Donaldson invariants in this regime. Notice that at every point of the quantum moduli space of vacua we have a different theory. The twisting of these theories gives precisely the effective TQFTs associated to Donaldson theory. In general, we will have two different kinds of theories.

At a generic point in the u -plane, the resulting theory is just the abelian version of Donaldson theory, as the effective description there includes only an abelian $N = 2$ vector multiplet. In the same way, these theories will simply describe the moduli space associated to abelian instantons on X .

Near the singularities, an $N = 2$ matter hypermultiplet, coupled to the abelian $N = 2$ vector multiplet, must be also considered. This hypermultiplet describes the additional light degree of freedom (monopole or dyon) near $u = \pm\Lambda^2$. To describe then the twisted theory at the singularities, we have to perform the twist of the $N = 2$ matter hypermultiplet. The hypermultiplet contains a complex scalar isodoublet $q = (q_1, q_2)$, fermions $\psi_{q\alpha}, \psi_{\tilde{q}\alpha}, \bar{\psi}_{q\dot{\alpha}}, \bar{\psi}_{\tilde{q}\dot{\alpha}}$, and a complex scalar isodoublet auxiliary field F_i . The fields $q_i, \psi_{q\alpha}, \bar{\psi}_{\tilde{q}\dot{\alpha}}$ and F_i are in the fundamental representation of the gauge group, while the fields $q^{i\dagger}, \psi_{\tilde{q}\alpha}, \bar{\psi}_{q\dot{\alpha}}$ and $F^{i\dagger}$ are in the conjugate representation. As the fields q_i are charged under the $SU(2)_I$ symmetry, after the twist they become positive-chirality spinors M_α . These fields are in fact sections of the bundle $S^+ \otimes L$, where L is the line bundle associated to the $U(1)$ gauge field, and the resulting twisted theory will not be well defined on a general four-manifold. However, it has been shown in ⁴¹ that the *dual* line bundle for the magnetic photon is not globally defined as a line bundle, but rather defines a Spin^c -structure. This means, on one hand, that L^2 is well defined as a line bundle, and on the other hand that the tensor product $S^+ \otimes L$ is also well defined. This tensor product bundle is called the (positive-chirality) complex spinor bundle associated to the Spin^c -structure defined by L . Once this subtlety has been understood, we can already read from the twisted action the equation for the section s . In this case, the configuration space for a given choice of the Spin^c -structure is

$$\mathcal{X} = \mathcal{A}_{L^2} \times \Gamma(X, S^+ \otimes L), \quad (24)$$

where the first factor refers to the $U(1)$ connections on the line bundle L^2 , and the second factor denotes the sections of the positive-chirality complex spinor bundle. The vector space \mathcal{F} is now,

$$\mathcal{F} = \Omega^{2,+}(X) \oplus \Gamma(X, S^- \otimes L), \quad (25)$$

where the second factor denotes the sections of the negative-chirality complex spinor bundle. The section s has then two components, and its zero locus is defined by the equations

$$\frac{1}{2}F_{\alpha\beta}^+ + i\overline{M}_{(\alpha}M_{\beta)} = 0,$$

$$D_{L^2}^{\dot{\alpha}\alpha} M_{\alpha} = 0. \quad (26)$$

These are the celebrated *Seiberg-Witten monopole equations*, and were introduced in reference³⁹. We have written them using spinor notation. $F_{\alpha\beta}^+$ denotes the self-dual part of the curvature associated to the connection A on L^2 . The $1/2$ factor arises because what appears in the action is the curvature of the non-globally defined bundle L . We also denoted $\overline{M}_{(\alpha} M_{\beta)} = \overline{M}_{\alpha} M_{\beta} + \overline{M}_{\beta} M_{\alpha}$. In the second equation, D_{L^2} denotes the Dirac operator associated to the positive-chirality complex spinor bundle. The term $\overline{M}M$ in the first equation comes from the D -terms of $N = 2$ QED. The solutions to the equations (26), modulo gauge transformations, form a compact moduli space of dimension

$$\dim \mathcal{M}_{\text{SW}} = \frac{1}{4}(x^2 - 2\chi - 3\sigma), \quad (27)$$

where $x = -c_1(L^2)$. From now on we will denote Spin^c -structures by this x (recall that line bundles can be completely classified by their first Chern class). One can define invariants just like in Donaldson theory, and when the dimension of the moduli space of abelian monopoles is zero, one can compute the partition function of the corresponding TQFT. The resulting invariant, associated to a Spin^c -structure x on X , is called the Seiberg-Witten invariant, and will be denoted by n_x . The Spin^c -structures x for which the corresponding Seiberg-Witten invariants is non-zero are called *basic classes*. They verify in particular $x^2 = 2\chi + 3\sigma$. The TQFT associated to the moduli problem in (26) was described in detail in reference²¹. The mathematical aspects of the Seiberg-Witten monopole equations have been reviewed in reference²⁶

Now we can try to evaluate the generating functional (23). If we do the path integral in terms of the effective description, one has to integrate over the u -plane and then perform the path integral for each of the different effective theories. Let us discuss carefully the steps in this computation:

1) First of all, if we assume (as we did before) that $b_2^+ > 1$, an important simplification occurs. Recall that, if $u \neq \pm\Lambda^2$, then the effective TQFT describes the moduli space of abelian instantons on X . But if $b_2^+ > 1$ there are no abelian solutions to the ASD equations, and this moduli spaces is empty. Therefore, the only contributions to the path integral come from the singularities.

2) Let's focus now on the monopole singularity at $u = \Lambda^2$. We know that the TQFT there describes the Seiberg-Witten monopoles, *i.e.*, the solutions to (26). But we must also compute VEVs of operators, and then we must write the observables of Donaldson-Witten theory in terms of operators of the effective, abelian theory. To obtain the structure of this expansion we will use the expansion in the untwisted, physical theory, together with the descent equations in the topological abelian theory⁴⁰. The descent equations for the abelian monopole theory can be found in ²¹. Near the monopole singularity, the u variable has the expansion

$$u(a_D) = \Lambda^2 + \left(\frac{du}{da_D}\right)_0 a_D + \text{higher order}, \quad (28)$$

where $(du/da_D)_0 = -2i\Lambda^{32}$, and “higher order” means operators of higher dimensions in the expansion. The field a_D corresponds to the field ϕ_D of the topological abelian theory²¹, while the gauge-invariant parameter u corresponds to the observable (19) (after complex conjugation). In terms of observables of the corresponding twisted theories, the expansion (28) reads

$$\mathcal{O} = \langle \mathcal{O} \rangle - \frac{1}{\pi} \langle V \rangle \phi_D + \text{higher order}, \quad (29)$$

where $\langle \mathcal{O} \rangle$, $\langle V \rangle$ are real c -numbers which should be related to the values of $u(0)$, $(du/da_D)_0$ in the untwisted theory. From the observable \mathcal{O} one can obtain the observable $\mathcal{O}^{(2)}$ by the descent procedure, and applying this procedure in the abelian TQFT to the r.h.s. of (29), we obtain:

$$\mathcal{O}^{(2)} = -\frac{1}{\pi} \langle V \rangle F + \text{higher order}, \quad (30)$$

where F is the dual electromagnetic field (associated to the magnetic monopole). In particular, taking into account that $x = -c_1(L^2) = -[F]/\pi$, where $[F]$ denotes the cohomology class of the two-form F , we finally obtain:

$$I(\Sigma) = \langle V \rangle (\Sigma \cdot x) + \text{higher order}, \quad (31)$$

where the dot denotes the pairing between 2-cohomology and 2-homology. From the point of view of TQFT, higher dimensional terms should not contribute in the expansion, because of the invariance of the theory under rescalings of the metric. Manifolds for which this is indeed true are called of *simple type*. It seems that all simply connected four-manifolds with $b_2^+ > 1$ are of simple type, according to our expectations.

We can now evaluate the correlation function in (23) in terms of a path integral in the low energy twisted theory, for the monopole singularity. First of all, we have to sum over the topological sectors in the abelian twisted theory, *i.e.*, over all the Spin^c -structures x . For manifolds of simple type, the expansion of the observables only includes c -numbers, as we can see in (29) and (31), and, possibly, contact terms for the operators $I(\Sigma)$ (see reference³⁸ for a discussion on this point). These terms give the intersection form of the manifold. Denoting $v = \sum_a \alpha_a I(\Sigma_a)$, the intersection form appears as $\gamma v^2 = \gamma \sum_{a,b} \alpha_a \alpha_b (\Sigma_a, \Sigma_b)$, where γ is a real number. As the operators are now c -numbers, the computation of the path integral just gives the partition function of the theory for each Spin^c -structure x . In particular, only the basic classes appear in the sum over topological sectors. We then obtain, for the first singularity,

$$C \exp(\gamma v^2 + \mu \langle \mathcal{O} \rangle) \sum_x n_x e^{\langle V \rangle v \cdot x}, \quad (32)$$

The constant C appears in the comparison of the macroscopic and microscopic path integrals, after fixing the respective normalizations. Some steps to determine it have been given in reference⁴¹.

3) We should now obtain the contribution from the dyon singularity at $u = -\Lambda^2$. But to do this we only need to implement the \mathbf{Z}_8 symmetry of the underlying $N = 2$ supersymmetric Yang-Mills theory, as this symmetry in the u -plane relates the two vacua (see reference³⁸). With our choice of twisting, and the correspondence between physical and twisted fields, we see that the \mathcal{O} operator has \mathcal{R} -charge 4, while the $I(\Sigma)$ operator has \mathcal{R} -charge 2, and they transform under the generator α of the \mathbf{Z}_8 symmetry as

$$\alpha \mathcal{O} \alpha^{-1} = -\mathcal{O}, \quad \alpha I(\Sigma) \alpha^{-1} = i I(\Sigma). \quad (33)$$

Recall that \mathbf{Z}_8 is the anomaly-free discrete subgroup of the $U(1)_{\mathcal{R}}$ symmetry. But on a curved manifold there is a gravitational contribution to the anomaly that can be computed in a standard way using the index theorem. The path integral measure around the singularity at $u = -\Lambda^2$ picks then a phase of the form

$$\exp\left(\frac{i\pi}{4} \dim \mathcal{M}_{ASD}\right) = i^\Delta, \quad (34)$$

where $\Delta = (\chi + \sigma)/4$, and we have used the fact that $\dim \mathcal{M}_{ASD}$ is an even integer.

Taking all this information into account, the correlation function of Donaldson theory is given by the sum of the monopole contribution, given in (32), and its \mathbf{Z}_8 transform, which gives the dyon contribution. The final result is

$$C\left(\exp(\gamma v^2 + \mu \langle \mathcal{O} \rangle) \sum_x n_x e^{\langle V \rangle v \cdot x} + i^\Delta \exp(-\gamma v^2 - \mu \langle \mathcal{O} \rangle) \sum_x n_x e^{-i \langle V \rangle v \cdot x}\right), \quad (35)$$

and the unknown universal constants can be obtained after comparing to explicit results in the mathematical literature,

$$C = 2^{1+(7\chi+11\sigma)/4}, \quad \gamma = 1, \quad \mathcal{O} = 2, \quad \langle V \rangle = 1. \quad (36)$$

This remarkable result for the Donaldson invariants in terms of Seiberg-Witten invariants was first obtained in reference³⁹, although in the case of Kähler manifolds it was previously obtained in reference³⁸, using exact results in $N = 1$ supersymmetry. The fact that Donaldson invariants have the structure shown in (35) for some 2-dimensional cohomology classes x is precisely the content of the Kronheimer-Mrowka structure theorem²⁰.

3. Non-Abelian Monopoles

So far we have discussed two different moduli problems in four-dimensional topology, the ASD instanton equations and the Seiberg-Witten monopole equations. There is a natural generalization of these moduli problems which is non-abelian and includes spinor fields. These are the *non-abelian monopole equations*, introduced in reference²² from the point of view of the Mathai-Quillen formalism and as a generalization of

Donaldson theory. They have been also considered in reference¹⁶ as well as in the mathematical literature^{29,33,31,12}.

To introduce these equations in the case of $G = SU(N)$ and the monopoles in the fundamental representation \mathbf{N} , consider a Riemannian four-manifold X together with an $SU(N)$ -bundle P and an associated vector bundle E in the fundamental representation. Suppose for simplicity that the manifold is Spin, and consider a section M_α^i of $S^+ \otimes E$. The non-abelian monopole equations read in this case,

$$\begin{aligned} F_{\alpha\beta}^{+ij} + i(\overline{M}_{(\alpha}^j M_{\beta)}^i - \frac{\delta^{ij}}{N} \overline{M}_{(\alpha}^k M_{\beta)}^k) &= 0, \\ (D_E^{\dot{\alpha}\alpha} M_\alpha)^i &= 0. \end{aligned} \tag{37}$$

These equations can be analyzed both from the mathematical and the physical point of view^{22,23,27,25,24}. We will here summarize some of the most salient features for the $SU(2)$ case:

1) The moduli space of non-abelian monopoles associated to (37) includes the moduli space of ASD instantons as a subset, and in fact the usual conditions to have a well-defined moduli problem (like the reducibility) are essentially the same as in Donaldson theory. The description of this moduli space can be made more precise in the case of Kähler manifolds^{22,29}.

2) The equations 37 arise in the twisting of $N = 2$ QCD with one massless hypermultiplet ($N_f = 1$)^{23,16,27}. They can be also defined on general four-manifolds using Spin^c -structures, as in the abelian case^{16,29,12,31}. It has been recently shown that, from the point of view of the twisted $N = 2$ QCD theory, the inclusion of Spin^c -structures corresponds to an extended twisting procedure associated to the gauging of the baryon number²⁵.

3) The moduli space of non-abelian monopoles has a natural $U(1)$ action which acts as a rotation on the monopole fields. The fixed points of this action are essentially the moduli space of ASD instantons and the moduli space of abelian Seiberg-Witten monopoles. This has opened the way to a mathematical proof of the equivalence of both theories using localization techniques^{31,29}, and some promising and concrete results in this direction have been recently obtained³⁰. From the point of view of the Mathai-Quillen formalism, the $U(1)$ action makes the bundle \mathcal{E} in (1) a $U(1)$ -equivariant bundle and one can obtain a general expression for the equivariant extension of the Thom form in this formalism²⁴. For the non-abelian monopole theory, the TQFT associated to this extension is precisely twisted $N = 2$ QCD with one massive flavour. It is tempting to think that the physics of the massive theory could shed some light on the localization problem.

4) Because of the equivalence of the non-abelian monopole theory and $N = 2$ supersymmetric QCD, one can compute the topological invariants associated to their moduli space using the techniques of the previous section and the Seiberg-Witten solution for the $N_f = 1$ theory²³. They can also be computed in the Kähler case

using exact results for superpotentials in $N = 1$ supersymmetry and some additional assumptions. In the $N_f = 1$ case, the quantum moduli space of vacua has three singularities related by a \mathbf{Z}_3 symmetry, and we have then three different contributions. The observables have the same structure as in the Donaldson-Witten theory (although they have a different geometrical content, as they implicitly include contributions from the matter fields). The effective description of the theory is again in terms of a dual $N = 2$ QED with one massless hypermultiplet, and the main result of this computation is that the invariants associated to the non-abelian monopole theory can be expressed in terms of Seiberg-Witten invariants. The generating function for manifolds with $b_2^+ > 1$ is²³:

$$\begin{aligned}
& \langle \exp(\sum_a \alpha_a I(\Sigma_a) + \mu \mathcal{O}) \rangle \\
&= C \left(\exp(\gamma v^2 + \mu \langle \mathcal{O} \rangle) \sum_x n_x \exp(\langle V \rangle v \cdot x) \right. \\
&+ e^{-\frac{\pi i}{6} \sigma} \exp\left(-e^{-\frac{\pi i}{3}}(\gamma v^2 + \mu \langle \mathcal{O} \rangle)\right) \sum_x n_x \exp(e^{-\frac{2\pi i}{3}} \langle V \rangle v \cdot x) \\
&+ \left. e^{-\frac{\pi i}{3} \sigma} \exp\left(-e^{\frac{\pi i}{3}}(\gamma v^2 + \mu \langle \mathcal{O} \rangle)\right) \sum_x n_x \exp(e^{-\frac{4\pi i}{3}} \langle V \rangle v \cdot x) \right).
\end{aligned} \tag{38}$$

where unknown constants appear as in the pure Donaldson-Witten case.

The result (38) suggests that moduli problems in four-dimensional topology can be classified in *universality classes* associated to the effective low-energy description of the underlying physical theory. One important question that should be addressed is how large is the set of moduli spaces which admit a description in terms of Seiberg-Witten invariants. It is very likely that in the search for this set new types of invariants will be found leading to new universality classes.

We will end comparing the situation found in four dimensions to the one in three. In the latter, it turns out that all the Chern-Simons invariants can be expressed in terms of Vassiliev invariants. These invariants play the role of the set of universality classes for this case. As in the four-dimensional case, the analysis of the TQFT from a perturbative and a non-perturbative point of view establishes their relation. In the four-dimensional case Seiberg-Witten invariants constitute the first universality class of a presumably large set of invariants. At present, the construction of this set is one of the most important challenges in TQFT.

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